Table 1 Relaxation time for various energies

Particle energy	Relaxation time, sec
10 ev	1.42×10^{-6}
36 ev	0.31
300 Kev	7.4
170 Mev	10^5 (27.7 hr)
3 Bev	7.4×10^6 (86 days)

other, they are repelled and can be scattered through large angles (90° or more) in a single encounter and thus escape from the containment device. The cross section σ_c for this close collision is given by (see Ref. 10, p. 88)

$$\sigma_c = e^4/4\,W^2 = 1.6 \times 10^{-14}\ \text{cm}^2/W^2$$

where W is the center of mass (or relative) kinetic energy of the interacting particles in electron volts. If t is the relaxation time or time per scatter to produce a substantial change in concentration, then $t = 1/n\,\sigma_c v$, where n is the concentration (assume 10^{15} particles per cm^3), and v is the particle speed. Since v is proportional to $W^{1/2}$, and since at thermal temperatures v is 22×10^5 cm/sec and W is 0.025 ev, it can be shown that $v = 13.9 \times 10^5\ W^{1/2}$, and therefore $t = 4.5 \times 10^{-8}\ W^{1/2}$. The relaxation time for various energies is shown in Table 1. The results indicate that retention, even for a day, would require very high energy in the hundreds of Mev.

Electron energy loss

At the high energies required for the antiprotons, the accompanying positive electrons would lose energy very rapidly by cyclotron radiation (proportional to the square of the energy) and, to a lesser extent, by bremsstrahlung radiation (proportional to the square root of the energy). This loss would have to be made up in effect by "cannibalizing" energy from the stored antiprotons and in a very short time no energy would be available for propulsion.

The significant fact is that at the high energies needed to reduce coulomb scattering of the antiprotons and permit their possible storage the concurrent losses of energy due to the positive electron radiation become unbearable.

Conclusions

Although antimatter would be a desirable propellant, its preparation in useful quantities is not currently possible, and no way can be visualized at present for storage of antiparticles. The high energies necessary to reduce coulomb scattering cause unbearable energy losses due to electron radiation. There appears to be no way out of this dilemma in the light of current understanding. Unforeseen basic developments in the next 8–10 years might alter the picture, but this is pure speculation. Future developments in basic plasma and fusion research should be watched, but no further feasibility work is justified for the present.

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Ergodic Theory and Certain Stochastic Satellite Problems

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I. Introduction

THE particular class of problems which motivated this report are as follows: We are given a certain initial configuration of satellites in their orbits. We are also given a particular "geometric"† event E of interest in connection with the performance of this system. A typical event E might be "the subsatellite point of at least one of the satellites in the system falls within a designated area on the surface of the earth," or "all of the satellites are simultaneously clustering around their respective southernmost points." We wish to define and calculate the probability of events of this nature and determine how this probability varies as a function of the initial condition. Transients, however, are not discussed. The results proven below were the consequence of an attempt to generalize and derive rigorously certain similar statements to be found on pp. 149 and 150 of Ref. 6. The bulk of the mathematical results utilized belong to those disciplines usually called "ergodic theory" and "diophantine approximation." Certain critical results referred to in the text are stated in the appendix for reference.

II. Mathematical Preliminaries

Let us assume we are given a system of n satellites in n (not necessarily distinct) orbit planes about a spherical nonrotating earth. Each of the j ($j = 1, \ldots, n$) satellites at time zero is assumed to be in a circular orbit of altitude h_j, have orbit inclination i_j, and have ascending nodal location Ω_j. The angular rate ω_j of the jth satellite is presumed constant. Let $\theta_j^*(t)$ be the true anomaly of satellite j at time t measured in the sense of motion around the orbit from the ascending node. We shall agree to use units of revolutions per unit time for ω_j and define $\theta_j(t)$ as $\theta_j^*(t)$ modulo I. More precisely, for any real number α, let $[\alpha]$ be the largest integer in α, so that $[\alpha] \leq \alpha < [\alpha] + 1$. Let $\{\alpha\}$ be the fractional part of α, i.e., $\{\alpha\} \equiv \alpha - [\alpha]$. Then $0 \leq \{\alpha\} < 1$. In general, we shall use the expressions "α modulo 1" and "$\{\alpha\}$" interchangeably.

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† The meaning of the adjective "geometric" will be explained more precisely in the following.

Let $\omega \equiv (\omega_1, \ldots, \omega_n)$, $\theta(t) \equiv [\theta_1(t), \ldots, \theta_n(t)]$, so that ω and $\theta(t)$ are vectors in Euclidean n-space. Then

$$\theta(t) = \{\theta(0) + \omega t\} \tag{1}$$

where ωt indicates multiplication of the vector ω by the scalar t, and $\{(\alpha_1, \ldots, \alpha_n)\} \equiv (\{\alpha_1\}, \ldots, \{\alpha_n\})$ for any vector $(\alpha_1, \ldots, \alpha_n)$ in n-space. Let $i = (i_1, \ldots, i_n)$, $\Omega = (\Omega_1, \ldots, \Omega_n)$. By a given system of n satellites we shall mean a system with given i and Ω.

The angular rate vector of the system is ω. The initial configuration of the system in "configuration space" is $\theta(0)$. The position (configuration) of the system at time t is $\theta(t)$.

A geometric event means an event whose occurrence for the given system is a function of $\theta(t)$ only. Now let $g_E(t) \equiv f[i, \Omega, \theta(t)]$ be the time function of some specific geometric event E, i.e., $g_E(t) = 1$ if E occurs for the given system at time t and $g_E(t) = 0$ otherwise. Now by definition i and Ω are fixed for our given system. Also by Eq. (1), $\theta(t)$ is a function of $\theta(0)$ and ω only. We highlight these facts by writing $g_E(t) = g_{E,\theta(0),\omega}(t)$ for the time function of the geometric event E for a system with given (fixed) i, Ω.

We must now define what we mean by the probability $P(E)$ of the occurrence of the event E for our system with given $\theta(0)$ and ω. We shall define $P(E)$ to be the expected fraction of the time that E occurs for our given system with given $\theta(0)$ and ω. In symbols,

$$P(E) \equiv \lim_{L\to\infty} \frac{1}{L} \int_0^L g_{E,\theta(0),\omega}(t)dt \tag{2}$$

Having made this definition, we face the problem of proving that the limit on the right-hand side of Eq. (2) exists. For the moment, we assume that it does exist and observe that $P(E)$ for our given system is a function of the (additional) initial conditions $\theta(0)$ and ω. Our problem is now to calculate $P(E)$ and to determine how $P(E)$ varies as a function of $\theta(0)$ and ω.

III. Mathematical Model

Let I be the interval [0, 1] on the real line. Let $I_n = I x I \ldots x I$ n times; otherwise stated I_n is the set of all $(x_1, \ldots, x_n)$ in n-space such that $0 \leq x_i < 1$ for $1 \leq i \leq n$. Now the set I_n, considered together with the Borel sets and Lebesgue measure μ, is a probability space.[4]

We shall confine our attention to (satellite systems, event E) pairs that permit the following mathematical model. The geometrical configuration of the system at time t depends upon the following.

Constants $\gamma_1, \ldots, \gamma_r$ (possibly a vacuous or null set)

Time-dependent parameters $\theta_1(t), \ldots, \theta_n(t)$ whose dependence on time may be expressed in the form $\theta(t) = \{\theta(0), +\omega t\}$, for a given constant vector $\omega = (\omega_1, \ldots, \omega_n)$, $\omega_i \neq 0$, where $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]$, and the $\{\ \}$ symbol indicates modulo 1 as previously noted. Note that $\theta(t)$ is in I_n.

Time-dependent parameters $\beta_1(t), \ldots, \beta_s(t)$

We also assume that the occurrence of the event E at time t depends only upon $\gamma_1, \ldots, \gamma_r$, and $\theta_1(t), \ldots, \theta_n(t)$, and not at all upon $\beta_1(t), \ldots, \beta_s(t)$. Since $\gamma_1, \ldots, \gamma_r$ are constant, this permits the mathematical abstraction of the event E (for the system with given $\gamma_1, \ldots, \gamma_r$) to be a fixed (measurable) subset E of I_n. Thus, for the given system, the event occurs at time t if and only if $\theta(t)$ is in E. Let ψ_E be the characteristic function of the set E. Then for α in I_n, $\psi_E(\alpha) = 1$, if α is in E, and $\psi_E(\alpha) = 0$ otherwise. It should now be clear that

$$g_{E,\theta(0),\omega}(t) = \psi_E[\theta(t)]$$

where the function on the left is the time function for the event E and has been defined in the foregoing.

IV. Some General Results and Their Interpretation

For any given $\omega = (\omega_1, \ldots, \omega_n)$ and fixed time t, consider the transformation T_t on I_n to I_n defined by

$$T_t\alpha \equiv \{\alpha + \omega t\} = \alpha + \{\omega t\} \text{ modulo } 1 \tag{3}$$

for any α in I_n. Then T_t is an invertible measure-preserving transformation of I_n onto itself. The so-called Individual Ergodic Theorem (Birkhoff)‡ may then be invoked in our context to prove

$$P[E, \theta(0), \omega] = \lim_{L\to 0} \frac{1}{L} \int_0^L \psi_E[T_t\theta(0)]dt \tag{4}$$

exists for almost every $\theta(0)$ and every ω. Considered as a function of $\theta(0)$, $P[E, \theta(0), \omega]$ is integrable and

$$\mu(E) = \int P[E, \theta(0), \omega]d\theta(0) \tag{5}$$

where $\mu(E)$ = Lebesgue measure of E

$$P[E, \theta(0), \omega] = P[E, \theta(0) + \omega t, \omega] \tag{6}$$

Note that (4) guarantees the existence of the right-hand side of Eq. (2) and renders meaningful the interpretation of the left-hand side of Eq. (4) as the expected fraction of the time the event E occurs. The interpretation of (5) is important and useful. It says that the number $\mu(E)$ is the average (or expected) value of $P[E, \theta(0), \omega]$ with respect to $\theta(0)$, i.e., $\mu(E)$ is $P(E)$ averaged with respect to the starting condition $\theta(0)$. It also says that the latter average is independent of ω. The statement (6) says that the expected fraction of the time that E occurs during a given trajectory does not depend upon the starting θ on that trajectory.

Stronger results than (4–6) are deducible from the preceding if I_n is indecomposable with respect to the set of transformations T_t mentioned previously. A measurable subset S of I_n is called invariant with respect to the set of transformations T_t if $s\ T_t$ is in S for all t given that s is in S. The space I_n will be called *metrically indecomposable* if the only invariant subsets have (Lebesgue) measure either 0 or 1. Assuming this we may assert that $P[E, \theta(0), \omega]$, considered as a function of $\theta(0)$, is a constant almost everywhere and

$$P[E, \theta(0), \omega] = \mu(E) \tag{7}$$

almost everywhere.

The implications of (7) are interesting. Observe that $\mu(E)$ answers the question, "Given a point drawn at random (uniformly distributed) from I_n, what is the chance that this point is in E?" The quantity $P[E, \theta(0), \omega]$ answers the question, "What is the expected fraction of the time that the event E will occur on a trajectory that starts at $\theta(0)$ and has movement variables ω?" Let us call $\mu(E)$ the "space" mean and $P[E, \theta(0), \omega]$ the "time" mean. Then, provided ω is such that the induced transformations T_t render I_n metrically indecomposable, we have: 1) the "time" mean is constant and has a value independent of $\theta(0)$ and ω, and 2) the "time" mean is equal to the "space" mean.

We close this section with the remark that I_n will be metrically indecomposable with respect to the T_t provided the components of ω are integrally independent. The latter means that no integers $K_1, \ldots, K_n$ not all zero may be found such that $K_1\omega_1 + \ldots + K_n\omega_n = 0$. Equivalently, the latter means that no ω_i is a rational linear combination of the remaining ω_j's. If the vector ω is chosen at random (in any natural sense), then its components will be integrally independent with probability 1.

‡ See Appendix, theorem 3. Note that (4) corresponds to the continuous case of the (A1) in theorem 3; that f^* converges almost every where; (5) follows from (A2); (6) from (A3); (7) from (A4), and (8) from (A5).

V. Examples

Suppose we consider a system of n satellites in circular polar orbits which have initial position $\theta(0)$ and angular rate vector ω. The event E shall mean that the satellites simultaneously cluster around the South Pole; more precisely, the event $E(\epsilon)$ occurring at time t means $|\theta_i(t) - \frac{3}{4}| < \epsilon$ for all $i = 1, \ldots, n$. Recall that $\theta_1 = \frac{3}{4}$ means that the ith satellite is $\frac{3}{4}$ of a revolution past the ascending node, i.e., at the South Pole. We shall now consider the following questions: does the event $E(\epsilon)$ ever occur?

If it does occur, what is the probability P that $E(\epsilon)$ occurs? The answer to the first question is that $E(\epsilon)$ almost always occurs. More precisely, if the components of ω are integrally independent (an event that almost always occurs), then the event $E(\epsilon)$ will occur regardless of the given $\theta(0)$. Again, assuming the integral independence of the components of ω, we obtain $P = (2\epsilon)^n$. This follows from (7), recognizing that the (subset) E in this case is $E = (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)x \ldots x(\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$ where $(\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$ is the open interval on the real line having the indicated end points. Thus, $\mu(E) = (2\epsilon)^n = P$. Note that P is independent of ω and $\theta(0)$, as long as the components of ω are integrally independent.

Suppose we consider a system of n satellites in coplanar circular orbits which have initial position $\theta(0)$ and angular rate vector ω. Suppose the event E means all n satellites are simultaneously in the angular strip $\alpha \leq \theta \leq \beta$. Then, as previously noted, if the components of ω are integrally independent, $P(E) = (\beta - \alpha)^n$. Thus, in this case, $P(E)$ is independent of the values of $\theta(0)$ and ω. Note that questions such as the distribution of $P(E)$ considered as a function of $\theta(0)$ for a given distribution of $\theta(0)$ are completely answered and subtended in the fact that $P(E)$ is constant and does not change as $\theta(0)$ is varied.

VI. Discrete Time Formulation

As before, for any given $\omega = (\omega_1, \ldots, \omega_n)$ and time τ, the transformation $T = T_\tau$ on I_n to I_n defined by $T_\tau\alpha \equiv \{\alpha + \omega\tau\}$ for any α in I_n is an invertible measure-preserving transformation of I_n onto itself. We imagine τ to be a discrete elementary time interval and observe that $T^n = T_{n\tau}$. The discrete analog of (4) is

$$P[E, \theta(0), \omega] = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi_E[T^j\theta(0)]$$

which converges for almost every $\theta(0)$ and every ω. Results (5) and (6) remain unchanged with the understanding that the t in (6) is restricted to be of the form $m\tau$ for integral m. A set S will be called *invariant under* T if Ts is in S for all s in S. The transformation T is called *ergodic* if the only measurable subsets of I_n invariant under T have measure either 0 or 1. If T is ergodic then the results (7) and (8) follow as before. The transformation T defined is ergodic if no integers $K_1, \ldots, K_n$ not all zero may be found such that $K_1\omega_1 + \ldots + K_n\omega_n$ is an integer. If the vector ω is chosen at random, then its components satisfy the foregoing requirement with probability 1.

Appendix

We shall say that $\omega_1, \ldots, \omega_n$ are *integrally independent* if $K_1\omega_1 + K_2\omega_2 + \ldots + K_n\omega_n = 0$ for integer K_i implies all $K_i = 0$. If $K_1\omega_1 + K_2\omega_2 + \ldots + K_n\omega_n =$ integer for integer K_i implies all $K_i = 0$, then we shall say that $\omega_1, \ldots, \omega_n$ are *integrally independent modulo 1*. It should be clear that if $\omega_1, \ldots, \omega_n$ are integrally independent mod 1, then they are integrally independent but not vice versa. Let $I = (0, 1)$ and $I_n = Ix \ldots xIn$ times. Let $0 \leq a_j < \beta_j \leq 1$ $(1 \leq j \leq n)$, and define $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, and $S(\alpha, \beta) = (\alpha_1, \beta_1)x(\alpha_2, \beta_2)x \ldots x(\alpha_n, \beta_n)$. Let v_i $(i = 1, \ldots, \ldots)$ be a sequence of vectors in I_n. Let $N(v, \alpha, \beta)$ be the number of the first N vectors $v_1, \ldots, v_N$ that are in $S(\alpha, \beta)$. Then we shall say that the v_i are *uniformly distributed* in I_n if

$$\lim_{N\to\infty} \frac{N(v, \alpha, \beta)}{N} = \prod_{j=1}^{n} (\beta_j - \alpha_j) = \text{volume of } S(\alpha, \beta)$$

for all α, β. As before, let $\{\alpha\}$ be the fractional part of the real number α.

Theorem 1[3]

Let $\omega_1, \ldots, \omega_n$ be integrally independent mod 1. Then the sequence of vectors $v_i = (\{i\omega_1\}, \{i\omega_2\}, \ldots, \{i\omega_n\})$ $(i = 1, 2, \ldots)$ is uniformly distributed in I_n.

Corollary 1: Let ω be irrational. Then the sequence of numbers $\{\omega\}, \{2\omega\}, \{3\omega\}, \ldots$, is uniformly distributed in $I = (0, 1)$.

Corollary 2: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be in I_n and suppose $\omega_1, \ldots, \omega_n$ are integrally independent mod 1. Then for any $\epsilon > 0$, there is a positive integer q such that $|\{q\omega_i\} - \alpha_i| < \epsilon$ for $i = 1, \ldots, n$. As noted previously, the set I_n, considered together with the Borel sets and Lebesgue measure μ, is a probability space. If T is a measurable transformation on I_n to I_n and E is a Borel set, a point x of E is called *recurrent with respect to E and T* if T^nx is in E for at least one positive integer n. A set E is called *invariant under T* if Tx is in E for all x in E. The transformation T is called *ergodic* if the only Borel subsets of I_n invariant under T have measure either 0 or 1.

Theorem 2[2,5,§]

If T is a measure-preserving transformation on I_n and E is a Borel set of positive measure, then almost every point of E is recurrent. Moreover, for almost every x in E, there are infinitely many positive integers n such that T^nx is in E. If $n(x)$ denotes for each x in E the least positive integer such that $T^{n(x)}x$ is in E, then $n(x)$ is defined almost everywhere in E. Then, if T is ergodic,

$$\frac{1}{\mu(E)} \int_E n(x)dx = \frac{1}{\mu(E)}$$

In other words, the average length of time that it takes a point of E to return to E is the reciprocal of the measure of E.

Theorem 3[1,2,¶]

If T is a measure-preserving transformation on I_n and if f is in L_1, then

$$\frac{1}{n} \sum_{j=0}^{n-1} f(T^jx) \text{ converges almost everywhere} \tag{A1}$$

The limit function f^* is integrable and satisfies the identities

$$\int f^*(x)dx = \int f(x)dx \tag{A2}$$

$$f^*(Tx) = f^*(x) \text{ almost everywhere} \tag{A3}$$

If in addition, T is ergodic, then

$$f^* \text{ is a constant (almost everywhere)} \tag{A4}$$

$$f^* = \int f(x)dx \tag{A5}$$

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§ This theorem may be called Poincare's recurrence and mean recurrence theorem.

¶ Following Halmos, we shall call this the individual ergodic theorem.

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Kinetic Theory Analysis of Diffusion of Discontinuity Plane of Flow Velocity

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PROBLEMS in which state variables (e.g., density, velocity, etc.) change considerably in a short range (comparable to mean free path) in a gas are of main interest in rarefied gasdynamics as well as those[1,3] in which the conditions of a solid boundary in a gas change in a short time (comparable to mean collision period of gas molecules). As a simple and fundamental example of the former phenomena, we here try to investigate how a plane of initial discontinuity in gas velocity diffuses on the basis of the Boltzmann equation with *B-G-K* model.[4]

At time $t = 0$,‡ the gas is assumed to have uniform velocity U in x direction in the region $y > 0$ and $-U$ in $y < 0$, uniform density ρ_0 and uniform temperature T_0, and to be in equilibrium in respective regions. The plane at $y = 0$ of initial discontinuity in gas velocity is released to diffuse for $t > 0$. We further assume that the initial velocity U is much less than the velocity of sound in the gas so that the fundamental equations as well as the initial and boundary conditions may be linearized. Then, we can show that the density and temperature of the gas remain constant for $t > 0$. More generally, we can treat the velocity in x direction independently of the density, temperature, and velocity in y direction in the problem where the initial density and temperature are also not uniform.§ Accordingly, our result given below may represent the field of velocity in x direction in this more general problem.

On the basis of the *B-G-K* model, the linearized kinetic equations become

$$\left.\begin{aligned} \partial\phi\ \partial t + v\ \partial\phi\ \partial y &= \lambda(-\phi + 2huq_x) \\ q_x &= \iiint_{-\infty}^{\infty} u\phi F d\mathbf{v} \\ F = (h/\pi)^{3/2}\exp\{-h\,(u^2 + v^2 + w^2)\} &\qquad h = m/2kT_0 \end{aligned}\right\} \quad (1)$$

where $(\rho_0/m)F(1 + \phi)$ is the velocity distribution function, q_x the x component of the gas velocity, m the mass of a molecule, $\mathbf{v} = (u, v, w)$ the molecular velocity, k the Boltzmann constant, and λ a constant (collision frequency) related to the mean free path l as follows: $l = 2/\{(\pi h)^{1/2}\lambda\}$. The initial condition is

$$\begin{aligned} \phi &= 2huU \qquad (y > 0) \\ &= -2huU \qquad (y < 0) \end{aligned} \qquad \text{at } t = 0 \qquad (2)$$

whereas the boundary conditions are

$$\begin{aligned} \phi &= 2huU \qquad (v < 0) \qquad \text{as } y \to \infty \\ &= -2huU \qquad (v > 0) \qquad \text{as } y \to -\infty \end{aligned} \qquad (3)$$

ϕ is continuous at $y = 0$. From (1–3) we obtain, after some reduction, the following integral equation for q_x:

$$\frac{\bar{q}_x}{U} = \frac{\pm 1}{s + \lambda}\left\{1 - \frac{2}{\pi^{1/2}} J_0[h^{1/2}(s + \lambda)|y|]\right\} + \lambda\left(\frac{h}{\pi}\right)^{1/2}\int_{-\infty}^{\infty}\frac{\bar{q}_x}{U} J_{-1}[h^{1/2}(s + \lambda)|y - y_0|]dy_0 \qquad (4)$$

where the bar over a letter indicates the Laplace transform $(t \to s)$, the upper sign holds for $y > 0$ and the lower for $y < 0$, and the functions J_n's are defined by the integral[5]

$$J_n(\xi) = \int_0^{\infty} \zeta^n \exp\left[-\zeta^2 - \frac{\xi}{\zeta}\right] d\zeta$$

For short times ($\lambda t \ll 1$), the state of affairs is expected to be close to the free molecular flow ($\lambda = 0$). For free molecular flow, the right-hand side of Eq. (4) degenerates into its first term, and we easily obtain the solution

$$\frac{q_x}{U} = \frac{2}{\pi^{1/2}}\int_0^{h^{1/2}y/t}\exp(-\zeta^2)d\zeta \qquad (5)$$

The solution for short times for our problem may be obtained by adding a perturbation to this result. Thus, we obtain

$$\frac{q_x}{U} \cong \frac{2h^{1/2}y}{\pi^{1/2}t}\,[1 + \lambda t\{2^{1/2}\log(2^{1/2} + 1) - 1\}] \qquad \left|\frac{h^{1/2}y}{t}\right| \ll 1 \qquad (6a)$$

$$\frac{q_x}{U} \cong 1 - \frac{t}{\pi^{1/2}h^{1/2}y}\left[1 - \frac{1}{2}\left(\frac{t}{h^{1/2}y}\right)^2 - \lambda t\left\{1 - \frac{3}{2}\left(\frac{t}{h^{1/2}y}\right)^2\right\}\right]\exp\left\{-\left(\frac{h^{1/2}y}{t}\right)^2\right\} \qquad \left|\frac{h^{1/2}y}{t}\right| \gg 1 \qquad (6b)$$

The effect of molecular collision at the initial stage appears in the following way. Molecules coming from the region $y > 0$ have lost some of their average velocity in x direction by collision with molecules that have emerged from $y < 0$, whereas molecules from $y < 0$ have obtained the average velocity by collision with molecules from $y > 0$. For $y > 0$, the latter effect dominates the former. Thus, the flow is less decelerated than the free molecular flow there. In other words, the diffusion (or mixing) is slowed down by molecular collisions. It is also noted that the result (6a) does not contain any nonanalytic term such as $(h^{1/2}y/t)\log(h^{1/2}y/t)$ in contrast to the case in which a solid boundary exists as in Rayleigh flow.[1,2] Figure 1 shows the distribution of gas velocity against $h^{1/2}y/t$ for $\lambda t = 0.4$ (the result for free molecular flow is also shown for comparison).

For $\lambda t \gg 1$, the field may be described by the classical result based on the Navier-Stokes equation. That is, the classical solution with kinematic viscosity $\nu = 1/2h\lambda$ (corresponding to *B-G-K* model)

$$\frac{q_x}{U} = \frac{2}{\pi^{1/2}}\int_0^{h^{1/2}\lambda y/(2\lambda t)^{1/2}}\exp(-\xi^2)\,d\xi \qquad (7)$$

may be seen to satisfy the integral equation (4) by direct substitution, if we assume t^{-1} and $h^{1/2}y/t$ small¶ and neglect these

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‡ t is time and x, y, z is the Cartesian coordinate system.

§ The proof may be given in quite a similar way as in Ref. 2. Nonuniformity must be small to assure the linearization.

¶ Further study is required to clear the detailed behavior for $y \gtrsim h^{-1/2}t$. The region $y \sim h^{-1/2}t$ is interesting especially in the case where pressure pulse (sound wave) travels into the gas as in Ref. 3, since the ridge of the sound pulse is advancing there.